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Tangent Lie algebroids

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Abstract. This paper shows that a Lie algebroid structure on a smooth vector bundle $A \xrightarrow{\pi} Q$ gives rise to a Lie algebroid structure on the bundle $TA \xrightarrow{T\pi} TQ$, called the tangent Lie algebroid. The analysis uses global arguments. A Lie algebroid A is equivalent to a certain Poisson structure on A^*, and the tangent bundle of any Poisson manifold has a tangent Poisson structure. The tangent Poisson structure on TA^* is then dualized to produce the tangent Lie algebroid structure on TA. Local calculations are used, and formulae for local brackets are given.

1. Introduction

Poisson brackets are central to the subject of Hamiltonian systems. Poisson brackets may be thought of as an algebra structure on some ring related to the ring of smooth functions $C^\infty(Q)$ on a smooth manifold Q. When the algebra exists on the ring of smooth functions itself, the manifold is said to have a Poisson structure determined by the Poisson bracket. A Poisson structure is equivalent to a bundle map $\pi : T^*Q \to TQ$ together with an integrability condition stating the Jacobi identity of the Lie bracket. A closed 2-form on a manifold Q determines a Poisson bracket on the ring of functions constant along the characteristic distribution of the 2-form; this is called a pre-symplectic structure on Q. A symplectic structure on a manifold is a pre-symplectic structure whose characteristic distribution is zero, i.e. the map $\pi : T^*Q \to TQ$ is invertible.

A Lie algebroid is a vector bundle over a manifold with a Lie bracket on sections, and with a bundle map to the tangent bundle, called the anchor map, that is also a Lie algebra homomorphism on sections. Lie algebroids are very closely related to Poisson structures. A Poisson structure on a manifold Q is a particular kind of Lie algebroid structure on the cotangent bundle T^*Q, with anchor map $\pi : T^*Q \to TQ$, whose integrability condition is that the anchor is a homomorphism on sections; the bracket for this algebroid is written down explicitly in section 3.

A Poisson structure on a manifold Q determines a special Poisson structure on the bundle TQ, so it is not surprising that a Lie algebroid structure on a bundle $A \to Q$ determines a Lie algebroid structure on the bundle $TA \to TQ$.

We now introduce some useful notation. If (q^i) are a system of local coordinates on the manifold Q, then the induced coordinates on TQ are given by $(q^i, \dot{q}^j)$. Next, suppose that $f \in C^\infty(Q)$; we denote by $\dot{f}$ the 'prolongation' of f to TQ, namely the 1-form $\mathrm{d}f$ viewed as a function on TQ. Locally $\dot{f}$ is given by

$$\dot{f} = \frac{\partial f}{\partial q}\dot{q} = f_{,i}\,\dot{q}^i\,.$$

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We adopt the convention that upper indices are attached to functions, while lower indices are attached to sections of bundles. Finally, by the linear functions on $A \to Q$, we mean the sections of the dual bundle $A^* \to Q$.

2. Algebroids and Poisson structures

We introduce some definitions and examples.

Definition 1. A Lie algebroid over a manifold Q is a vector bundle $A \xrightarrow{\pi} Q$ together with a Lie algebra bracket on sections and a bundle map $A \xrightarrow{\rho} TQ$, resulting in the following commuting diagram, which together satisfy two conditions:

$$\begin{array}{ccc} A & \xrightarrow{\rho} & TQ \\ {\scriptstyle\pi}\downarrow & \swarrow{\scriptstyle\tau_Q} & \\ Q & & \end{array} \tag{2.1}$$

(i) The map ρ is a Lie algebra homomorphism between sections of A and vector fields on Q (i.e. sections of $TQ \xrightarrow{\tau_P} Q$),

(ii) The following Leibniz rule holds for a function $f \in C^\infty(Q)$ and sections X and Y of $A \xrightarrow{\pi} Q$:

$$[X, fY] = f[X, Y] + (\rho(X)f)Y\,. \tag{2.2}$$

Definition 2. A Poisson structure on a manifold Q is a Lie algebra bracket $\{\ ,\ \}$ on the ring of smooth functions $C^\infty(Q)$, which satisfies the Leibniz identity:

$$\{f, gh\} = g\{f, h\} + \{f, g\}h\,. \tag{2.3}$$

Equivalently, a Poisson structure on Q is a skew-symmetric bundle map $\pi : T^*Q \to TQ$ whose associated 2-tensor (also denoted by π) determines the bracket on functions by

$$\{f, g\} = \langle \pi | \mathrm{d}f \wedge \mathrm{d}g \rangle\,. \tag{2.4}$$

The components of the tensor π are given by $\pi^{ij} = \{q^i, q^j\}$ in local coordinates (q^i). The Jacobi identity for the bracket is then

$$\pi,_r^{ij}\pi^{rk} + \pi,_r^{jk}\pi^{ri} + \pi,_r^{ki}\pi^{rj} = 0\,. \tag{2.5}$$

The left-hand side of this equation is the ijk-component of the tensor $[\pi, \pi]$—the Schouten bracket of π with itself. Thus a Poisson structure may be defined as a bi-vector field π such that the Schouten bracket $[\pi, \pi]$ vanishes identically.

According to the definition a Lie algebroid structure on $T^*Q \to Q$ is a bracket $\{\alpha, \beta\}$ on 1-forms $\alpha, \beta \in \Gamma(T^*Q)$. As indicated in the introduction, there is such a Lie algebroid bracket on 1-forms which agrees with the Poisson bracket on exact 1-forms.

3. Algebroids

It is well known that linear Poisson structures correspond to dual Lie algebras. For a review of these and other facts, see [16]. In a linear Poisson structure, the Poisson bracket on linear functions on the dual Lie algebra is given by the Lie algebra bracket on the Lie algebra elements. We now extend this relationship.

Let $[\ ,\ ]$ be an algebroid bracket on the bundle $A \xrightarrow{\pi} Q$. Sections of A are linear functions on A^*, and hence the algebroid bracket may be thought of as a Lie bracket on the linear functions on A^*. We may go further and define a Poisson bracket $\{\ ,\ \}$ on A^* as follows. Let $\tilde{\mu}, \tilde{\nu}$ denote the linear functions on A^* determined by $\mu, \nu \in \Gamma(A)$, and let $f, g \in C^\infty(Q)$ be functions on the base. Then the relations

$$\{\tilde{\mu}, \tilde{\nu}\} = \widetilde{[\mu, \nu]} \tag{3.1}$$

$$\{\tilde{\mu}, f \circ \pi\} = (\rho(\nu) f) \circ \pi \tag{3.2}$$

$$\{f \circ \pi, g \circ \pi\} = 0 \tag{3.3}$$

determine a Poisson structure on A^*; see [3] and [4]. That the Jacobi identity for $\{\ ,\ \}$ is satisfied for linear functions follows because A is an algebroid, and it is satisfied for terms mixing linear functions with functions on the base because ρ is a Lie algebra homomorphism (see [3]). We now show that any Poisson structure on a vector bundle, whose linear functions form a subalgebra, is dual to a Lie algebroid.

Theorem 1. The bundle $\pi : A \to Q$ is an algebroid on Q with anchor $\rho : A \to TQ$ if and only if the dual bundle $A^* \to Q$ is a Poisson manifold whose linear functions form a Lie subalgebra.

Proof. Suppose we have a Poisson bracket $\{\ ,\ \}$ on $A^* \to Q$ such that the linear functions form a subalgebra. We may define an anchor $\rho : A \to TQ$ by

$$\rho(\mu) = \{\mu,\ \}. \tag{3.4}$$

To prove that ρ is an anchor we must show that ρ comes from a bundle map, i.e.

$$\rho(f\mu) = f\rho(\mu) \quad \text{whenever} \quad f \in C^\infty(Q) \quad \text{and} \quad \mu \in \Gamma(A). \tag{3.5}$$

This is established by successive application of the Leibniz rule for the Poisson bracket. Let $\mu, \nu \in \Gamma(A)$, and let $f, g \in C^\infty(Q)$; we may think of f and g as functions on A which are constant on fibres (i.e. we identify them with $f \circ \pi, g \circ \pi \in C^\infty(A)$). Now $\{\mu, f\nu\} = f\{\mu, \nu\} + \nu\{\mu, f\}$ must be linear, so functions of the form $\{\mu, f\}$ must be constant on fibres. This means that $\{f\mu, g\}$ is constant on fibres, so $\{f\mu, g\} = f\{\mu, g\} + \mu\{f, g\}$ is constant on fibres; therefore $\{f, g\} = 0$ and $\{f\mu, g\} = f\{\mu, g\}$. Thus we have $\rho(f\mu) = f\rho(\mu)$ and ρ is a bundle map. Finally, identifying μ with $\tilde{\mu}$ we obtain the derivation law

$$[\mu, f\nu] = f\{\mu, \nu\} + \nu\{\mu, f\} \tag{3.6}$$

$$= f[\mu, \nu] + (\rho(\mu) f)\nu. \tag{3.7}$$

Therefore ρ is an anchor and we have an algebroid bracket. □

4. Tangent Poisson structures

As stated in the introduction, when a bundle map $\pi : T^*Q \to TQ$ makes Q into a Poisson manifold, there is a Lie algebroid structure on T^*Q whose anchor is π. We now give this algebroid bracket, and determine as a consequence a Poisson structure on TQ induced by π; for a proof that the following bracket together with π form a Lie algebroid see [8].

Example 4.1. Given a Poisson structure π on Q, there is an algebroid structure on T^*Q with anchor map π given by

$$\{\alpha, \beta\} = \mathcal{L}_{\pi\alpha}\beta - \mathcal{L}_{\pi\beta}\alpha - \mathrm{d}\langle\pi|\alpha \wedge \beta\rangle \tag{4.1}$$

where $\mathcal{L}$ is the Lie derivative on 1-forms $\alpha, \beta \in \Gamma(T^*Q)$; see [3, 7, 12] and references therein.

Applying the homotopy formula $\mathcal{L}_\xi \theta = \xi \rfloor \mathrm{d}\theta + \mathrm{d}(\xi \rfloor \theta)$, we get

$$\{\alpha, \beta\} = \pi\alpha \rfloor \mathrm{d}\beta - \pi\beta \rfloor \mathrm{d}\alpha + \mathrm{d}\langle \pi | \alpha \wedge \beta \rangle . \tag{4.2}$$

Note that for exact 1-forms $\alpha = \mathrm{d}f$ and $\beta = \mathrm{d}g$ this gives us

$$\{\mathrm{d}f, \mathrm{d}g\} = \mathrm{d}\{f, g\} . \tag{4.3}$$

If we choose local coordinates (q^i) on Q, so that π has local components $\pi^{ij} = \{q^i, q^j\}$, then this algebroid structure on T^*Q is determined locally by

$$\{\mathrm{d}q^i, \mathrm{d}q^j\} = \mathrm{d}\{q^i, q^j\} \tag{4.4}$$
$$= \mathrm{d}\pi^{ij} \tag{4.5}$$
$$= \pi,_k^{ij}\ \mathrm{d}q^k \tag{4.6}$$

in other words the structure functions of the Lie algebroid are given locally by $\pi,_k^{ij}$ and the anchor components are given by π^{ij}.

The previous example shows that T^*Q has a Lie algebroid structure whenever Q has a Poisson structure. By theorem 1, it follows that the dual bundle to T^*Q, namely TQ, is a Poisson manifold whose linear functions form a subalgebra.

Theorem 2. If Q has a Poisson structure given locally by π^{ij} in the coordinates (q^i), then TQ has a Poisson structure in tangent coordinates $(q^i, \dot{q}^j)$ given by the relations

$$\{\dot{q}^i, \dot{q}^j\} = \pi,_k^{ij}\ \dot{q}^k \tag{4.7}$$
$$\{\dot{q}^i, q^j\} = \pi^{ij} \tag{4.8}$$
$$\{q^i, \dot{q}^j\} = \pi^{ij} \tag{4.9}$$
$$\{q^i, q^j\} = 0 . \tag{4.10}$$

Proof. This is an application of (2.1)–(2.3). □

Note that the linear functions do indeed form a subalgebra. This induced Poisson structure on TQ is called the tangent Poisson structure; see [2–4]. For alternative methods of realizing the tangent Poisson structure, see [2, 5].

5. Tangent Lie algebroids

5.1. The swap map

Let $A \xrightarrow{\pi} Q$ be a Lie algebroid, and let $p : A^* \times_Q A \to \Re$ be the natural pairing $p(u, a) = \langle u | a \rangle$. We may take tangents to get $Tp : TA^* \times_{TQ} TA \to \Re \times \Re$; locally this may be written as $Tp((u, \dot{u}), (a, \dot{a})) = (\langle u | a \rangle , \langle u | \dot{a} \rangle + \langle \dot{u} | a \rangle)$. We project onto the second factor, and denote the result

$$\langle\!\langle (u, \dot{u}) \mid (a, \dot{a}) \rangle\!\rangle = \langle u | \dot{a} \rangle + \langle \dot{u} | a \rangle\ .$$

This is a non-degenerate pairing $TA^* \times_{TQ} TA \to \Re$, and therefore we get an isomorphism $TA \to (TA^*)^*$ given locally by

$$(a, \dot{a}) \mapsto \langle\!\langle\ \cdot\ \mid (a, \dot{a})\ \rangle\!\rangle\ .$$

Theorem 3. The effect of the identification of TA with $(TA^*)^*$ given by the tangent of the natural pairing is to 'swap' a and $\dot{a}$.

Proof. The tangent pairing given above may be written as

$$\langle\langle (u,\dot{u}) \mid (a,\dot{a}) \rangle\rangle = [\,u \quad \dot{u}\,] \begin{bmatrix} \dot{a} \\ a \end{bmatrix} \tag{5.1}$$

so that $(a,\dot{a}) \in TA$ becomes identified with $(\dot{a},a) \in (TA^*)^*$. □

5.2. *The algebroid structure on TA*

In this section we exploit theorem 1 several times, by replacing Lie algebroids with Poisson structures and vice versa. We begin with a Lie algebroid $A \to Q$, so that A^* is a Poisson manifold. It follows that TA^* is again a Poisson manifold, with the tangent Poisson structure (a fact which also followed from theorem 1). Moreover, when viewed as a bundle $TA \to TQ$, its linear functions form a subalgebra. Therefore, by theorem 1, the bundle dual to $TA^* \to TQ$, namely $(TA^*)^* \to TQ$, is again a Lie algebroid. Thus whenever A is a Lie algebroid, there is an induced Lie algebroid structure on the bundle $(TA^*)^* \to TQ$. Finally, by using the swap map to identify TA with TA^*, we get an algebroid structure on $TA \to TQ$.

We now determine the local representation of the induced algebroid structure on $(TA^*)^* \to TQ$. Choose a local trivialization of the bundle $A \to Q$, i.e. local coordinates (q^i, a^j), where the a^js are linear functions on A, determined by a local basis of sections $a_j \in \Gamma(A^*)$.

Our choice of coordinates and local trivialization of A determines functions giving us the bracket and anchor of A locally:

$$[a_i, a_j] = c_{ij}^k\, a_k \tag{5.2}$$

$$\rho(a_i) = \rho_i{}^j \frac{\partial}{\partial q^j}\,. \tag{5.3}$$

Then according to (3.1)–(3.3) the Poisson structure on the dual bundle A^* is given by

$$\{a_i, a_j\} = c_{ij}^k\, a_k \tag{5.4}$$

$$\{a_i, q^j\} = \rho_i{}^j \tag{5.5}$$

$$\{q^i, q^j\} = 0\,. \tag{5.6}$$

We now compute the tangent Poisson structure on TA^* in the coordinates $(q^i, a_j, \dot{q}^i, \dot{a}^j)$ on TA induced by the local trivialization (q^i, a_j) on A.

Theorem 4. The tangent Poisson structure on TA^* is given by

$$\{\dot{a}_i, \dot{a}_j\} = \dot{c}_{ij}^k a_k + c_{ij}^k \dot{a}_k \tag{5.7}$$

$$\{\dot{a}_i, a_j\} = c_{ij}^k a_k \tag{5.8}$$

$$\{a_i, \dot{a}_j\} = c_{ij}^k a_k \tag{5.9}$$

$$\{\dot{a}_i, q^j\} = \rho_i^j \tag{5.10}$$

$$\{a_i, \dot{q}^j\} = \rho_i^j \tag{5.11}$$

$$\{\dot{a}_i, \dot{q}^j\} = \dot{\rho}_i^j \tag{5.12}$$

with all remaining brackets zero.

Proof. This is an application of (4.7)–(4.10). It is straightforward to check the Jacobi identity for these brackets, although this is guaranteed by theorem 1. □

Note that the linear functions of $TA^* \to TQ$ form a subalgebra under this tangent bracket. It follows that the dual bundle $(TA^*)^* \to TQ$ is an algebroid.

Theorem 5. The tangent Poisson structure on TA^* induces an algebroid structure on the bundle $(TA^*)^* \to TQ$ which is given locally by

$$[\dot{a}_i, \dot{a}_j] = \dot{c}_{ij}^k a_k + c_{ij}^k \dot{a}_k \tag{5.13}$$

$$[\dot{a}_i, a_j] = c_{ij}^k a_k \tag{5.14}$$

$$[a_i, \dot{a}_j] = c_{ij}^k a_k \tag{5.15}$$

$$[a_i, a_j] = 0 \tag{5.16}$$

$$\rho(a_i) = \rho_i{}^j \frac{\partial}{\partial \dot{q}^j} \tag{5.17}$$

$$\rho(\dot{a}_i) = \rho_i{}^j \frac{\partial}{\partial q^j} + \dot{\rho}_i{}^j \frac{\partial}{\partial \dot{q}^j}. \tag{5.18}$$

Proof. We have re-written as a Lie algebroid the Poisson bracket of theorem 4. □

In shorthand we have the brackets

$$[\dot{a}, \dot{a}] = \dot{c}a + c\dot{a} \tag{5.19}$$

$$[\dot{a}, a] = ca \tag{5.20}$$

$$[a, a] = 0 \tag{5.21}$$

with anchor $\rho(a, \dot{a}) = (\rho\dot{a}, \rho a + \dot{\rho}\dot{a})$.

Consider for the moment the Lie algebroid $A \xrightarrow{\pi} Q$, and with anchor ρ, as the commuting diagram

$$\begin{array}{ccc} A & \xrightarrow{\rho} & TQ \\ \pi\downarrow & \swarrow \tau_Q & \\ Q & & \end{array} \tag{5.22}$$

Taking tangents we get the diagram

$$\begin{array}{ccc} TA & \xrightarrow{T\rho} & TTQ \\ T\pi\downarrow & \swarrow T\tau_Q & \\ TQ & & \end{array} \tag{5.23}$$

and applying natural involution $\sim: TTQ \to TTQ$ and the identity $T\tau_{TQ} = \sim \circ \tau_{TQ}$ we get the diagram

$$\begin{array}{ccccc} TA & \xrightarrow{T\rho} & TTQ & \xrightarrow{\sim} & TTQ \\ T\pi\downarrow & T\tau_Q \swarrow & & \swarrow \tau_{TQ} & \\ TQ & & & & \end{array} \tag{5.24}$$

and hence we have this commutative diagram and the following theorem:

$$\begin{array}{ccc} TA & \xrightarrow{\sim \circ T\rho} & TTQ \\ T\pi\downarrow & \swarrow \tau_{TQ}. & \\ TQ & & \end{array} \tag{5.25}$$

Theorem 6. The Lie algebroid $A \xrightarrow{\pi} Q$, with anchor $\rho_A : A \to TQ$, induces a tangent Lie algebroid structure on $TA \xrightarrow{T\pi} TQ$, with anchor map $\rho_{TA} = \sim \circ T\rho_A$, given by

$$[a_i, a_j] = \dot{c}_{ij}^k \, \dot{a}_k + c_{ij}^k \, a_k \tag{5.26}$$

$$[\dot{a}_i, a_j] = c_{ij}^k \, \dot{a}_k \tag{5.27}$$

$$[a_i, \dot{a}_j] = c_{ij}^k \, \dot{a}_k \tag{5.28}$$

$$[\dot{a}_i, \dot{a}_j] = 0 \tag{5.29}$$

$$\rho(\dot{a}_i) = \rho_i{}^j \, \frac{\partial}{\partial \dot{q}^j} \tag{5.30}$$

$$\rho(a_i) = \rho_i{}^j \, \frac{\partial}{\partial q^j} + \dot{\rho}_i{}^j \, \frac{\partial}{\partial \dot{q}^j} \,. \tag{5.31}$$

Proof. We have simply written down the algebroid bracket on $(TA^*)^*$ and then applied the swap map to get this bracket and anchor on TA.

Although no further proof is needed, it is interesting to carry out a direct verification that the brackets and anchor above indeed determine a Lie algebroid on TA. For example, we check that ρ_{TA} is a homomorphism on sections.

The necessary calculations may be made directly. However, we use a method which requires some tools. First, recall that a vector field $X = \xi^i \partial/\partial q^i$ on Q has the so-called tangent lift $\dot{X} = \xi^i \partial/\partial q^i + \dot{\xi}^i \partial/\partial \dot{q}^i$, an invariantly defined vector field on TQ. With X viewed as a section of the tangent bundle $X : Q \to TQ$, we have

$$\dot{X} = \sim \circ TX \,. \tag{5.32}$$

Also, X has the so-called vertical lift $\hat{X} = \xi^i \partial/\partial \dot{q}^i$. We have the following lemma.

Lemma 1. Bracket relations among lifted vector fields are

$$\left[\dot{X}, \dot{Y}\right] = [X, Y]^{\cdot} \tag{5.33}$$

$$\left[\dot{X}, \hat{Y}\right] = [X, Y]^{\wedge} \tag{5.34}$$

$$\left[\hat{X}, \hat{Y}\right] = 0 \,. \tag{5.35}$$

With this notation, it is interesting to note that

$$\rho_{TA}(a_i) = [\rho_A(a_i)]^{\cdot} . \tag{5.36}$$

We now carry out the direct verification that ρ_{TA} is a homomorphism on sections:

$$\rho_{TA}([a_i, a_j]) = \rho_A\left(\dot{c}_{ij}^k \dot{a}_k + c_{ij}^k a_k\right) \tag{5.37}$$

$$= \dot{c}_{ij}^k \rho_k^r \frac{\partial}{\partial \dot{q}^r} + c_{ij}^k \rho_k^r \frac{\partial}{\partial q^r} + c_{ij}^k \dot{\rho}_k^r \frac{\partial}{\partial \dot{q}^r} \tag{5.38}$$

$$= c_{ij}^k \rho_k^r \frac{\partial}{\partial q^r} + (c_{ij}^k \rho_k^r)^{\cdot} \frac{\partial}{\partial \dot{q}^r} \tag{5.39}$$

$$= \left(c_{ij}^k \rho_k^r \frac{\partial}{\partial q^r}\right)^{\cdot} \tag{5.40}$$

$$= \left(\rho_A[a_i, a_j]\right)^{\cdot} \tag{5.41}$$

$$= \left[\rho_A(a_i), \rho_A(a_j)\right]^{\cdot} \tag{5.42}$$

$$= \left[\rho_A(a_i)^{\cdot}, \rho_A(a_j)^{\cdot}\right] \tag{5.43}$$

$$= \left[\rho_{TA}(a_i), \rho_{TA}(a_j)\right] . \tag{5.44}$$

We also have

$$\rho_{TA}([\dot{a}_i, a_j]) = \rho_{TA}(c_{ij}^k \dot{a}_k) \tag{5.45}$$

$$= c_{ij}^k \rho_k^r \frac{\partial}{\partial \dot{q}^r} \tag{5.46}$$

$$= \left(\rho_A[a_i, a_j]\right)\hat{}. \tag{5.47}$$

On the other hand we have

$$[\rho_{TA}(\dot{a}_i), \rho_{TA}(\dot{a}_j)] = \left[\rho_i^r \frac{\partial}{\partial \dot{q}^r}, \rho_j^s \frac{\partial}{\partial q^s} + \dot{\rho}_j^s \frac{\partial}{\partial \dot{q}^s}\right] \tag{5.48}$$

$$= \rho_i^r \rho_j^s{}_{,r} \frac{\partial}{\partial \dot{q}^s} - \rho_j^r \rho_i^s{}_{,r} \frac{\partial}{\partial \dot{q}^s} \tag{5.49}$$

$$= [\rho_A(a_i), \rho_A(a_j)]\hat{} \tag{5.50}$$

and therefore $\rho_{TA}([\dot{a}_i, a_j]) = [\rho_{TA}(\dot{a}_i), \rho_{TA}(\dot{a}_j)\ ]$. □

6. Examples

Example 6.1. We have already seen that a Poisson structure $\pi : T^*Q \to TQ$ is really an algebroid structure on $T^*Q \to Q$ with anchor π, i.e. an algebroid with the following commuting diagram:

$$\begin{array}{ccc} T^*Q & \xrightarrow{\pi} & TQ \\ {\scriptstyle \pi_Q}\downarrow & \swarrow {\scriptstyle \tau_Q} & \\ Q . & & \end{array} \tag{6.1}$$

Taking tangents we get

$$\begin{array}{ccc} TT^*Q & \xrightarrow{T\pi} & TTQ \\ {\scriptstyle T\pi_Q}\downarrow & \swarrow {\scriptstyle T\tau_Q} & \\ TQ . & & \end{array} \tag{6.2}$$

Now suppose that the tangent Poisson structure on TQ is given by the bundle map

$$\widetilde{T\pi} : T^*TQ \to TTQ .$$

Then $\widetilde{T\pi} = \sim \circ T\pi \circ \alpha^{-1}$ where $\sim$: $TTQ \to TTQ$ is the natural involution, and $\alpha : TT^*Q \to T^*TQ$ is a canonical involution (see [4, 5, 15]). The two bundle projections of TTQ are related by $\tau_{TQ} = T\tau_Q \circ \sim$. Using these facts we find that the tangent Poisson structure on TQ is given by the algebroid

$$\begin{array}{ccc} T^*TQ & \xrightarrow{\widetilde{T\pi}} & TTQ \\ {\scriptstyle \pi_{TQ}}\downarrow & \swarrow {\scriptstyle \tau_{TQ}\,.} & \\ TQ . & & \end{array} \tag{6.3}$$

To summarize, when Q is a Poisson manifold we have an algebroid structure on T^*Q, with a tangent algebroid structure on TT^*Q. The tangent Poisson structure on TQ gives rise to an algebroid structure on T^*TQ. These algebroid structures are intertwined by the canonical involution $TT^*Q \overset{\alpha}{\cong} T^*TQ$.

Example 6.2. An integrable Dirac structure on a manifold Q is a Lie algebroid structure on a bundle $L \to Q$, giving rise to a degenerate smooth foliation of Q by leaves equipped with degenerate closed 2-forms. Poisson and pre-symplectic structures are special cases of Dirac structures. We briefly review the facts here, (see [3, 4] for details; see [7, 8] for infinite dimensional examples and applications).

A Dirac bundle over Q is defined to be a sub-bundle $L \subset TQ \oplus T^*Q$ which is maximally isotropic under the natural pairing $\langle (X,\omega) \mid (Y,\mu)\rangle_+ = \frac{1}{2}(\langle X|\mu\rangle + \langle Y|\omega\rangle)$; in the Poisson case (respectively pre-symplectic case), the Dirac bundles are graphs of the bundle maps $\pi : T^*Q \to TQ$ (respectively $\Omega : TQ \to T^*Q$). Isotropy of L under the pairing $\langle\ ,\ \rangle_+$ means that for any $(X,\omega), (Y,\mu) \in \Gamma(L)$ we have $\omega(Y) = -\mu(X)$, thus generalizing the skew symmetry of the bundle maps in the Poisson and pre-symplectic cases.

A bracket is defined on Dirac bundles by

$$[(X,\omega),(Y,\mu)] = ([X,Y], \mathcal{L}_X\mu - \mathcal{L}_Y\omega + \mathrm{d}(\omega(Y)))\,. \tag{6.4}$$

Compare this bracket that induced on 1-forms by a Poisson structure in (4.2). This bracket is in general *not* a Lie algebroid bracket; it becomes a Lie algebroid bracket when an integrability condition holds. Consider the function $T_L : L \otimes L \otimes L \to \Re$ given by

$$\begin{aligned} T_L((X,\omega)\otimes(Y,\mu)\otimes(Z,\eta)) &= \langle([X,Y], \mathcal{L}_X\mu - \mathcal{L}_Y\omega + d(\omega(Y))) \mid (Z,\eta)\rangle_+ \\ &= \tfrac{1}{2}\{X\mu(Z) + Z\eta(X) + Z\omega(Y) + \mathrm{d}\omega(Y,Z) + \mathrm{d}\mu(Z,X) + \mathrm{d}\eta(X,Y)\}\,. \end{aligned} \tag{6.5}$$

The isotropy of L makes T_L into a 3-tensor on the vector bundle L. If $T_L = 0$ the Dirac structure L is called an integrable Dirac structure.

Theorem 7. (See [4].) The bracket of equation 6.4 is a Lie algebroid bracket on L if and only if $T_L = 0$. The anchor map is projected onto TQ.

If $T_L = 0$, then the distribution $\rho(L) \subset TQ$ obtained by projection is an integrable distribution (of possibly non-constant rank), whose leaves have closed 2-forms.

The special cases of integrable Dirac structures which we have already cited are Poisson manifolds, where the leaves have non-degenerate closed 2-forms, and pre-symplectic structures, where there is only one leaf. Integrable Dirac structures were shown to have tangent lifts in [4].

Example 6.3. A Lie algebra $\mathcal{G}$ is a Lie algebroid over a point. It follows that its structure functions are constant; they are just the structure constants of the Lie algebra. Therefore the terms $\dot{c}^k_{ij} = 0$. The equations for the tangent algebroid structure now become

$$[a_i, a_j] = c^k_{ij}\, a_k \tag{6.6}$$

$$[\dot{a}_i, a_j] = c^k_{ij}\, \dot{a}_k \tag{6.7}$$

$$[a_i, \dot{a}_j] = c^k_{ij}\, \dot{a}_k \tag{6.8}$$

$$[\dot{a}_i, \dot{a}_j] = 0\,. \tag{6.9}$$

These are the Lie algebra brackets for the semi-direct product of $\mathcal{G}$ with itself.

Example 6.4. Suppose that we are given $\mathcal{G}$ as the Lie algebra of a Lie group G. Then we may form the tangent Lie group TG, whose algebra is isomorphic to the semi-direct product $\mathcal{G}Ⓢ\mathcal{G}$, which as we saw in the previous example, is the tangent Lie algebroid. Therefore, in this case the tangent Lie algebroid is the algebroid of the tangent group.

In the same way, if A is the algebroid of a Lie groupoid G, then we may form the tangent groupoid TG. It is shown as theorem 7.1 in [11] that the tangent algebroid TA is the algebroid of the tangent groupoid TG.

The author wishes to note that an invariant form of the construction of section 4.2, namely 'A algebroid $\Rightarrow$ A^* Poisson $\Rightarrow$ TA^* tangent Poisson $\Rightarrow$ $(TA^*)^*$ algebroid $\Rightarrow$ TA tangent algebroid', has been carried out independently by McKenzie and Ping Xu in [11] using invariant methods.

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